EUROPEAN CENTRAL BANK

EUROSYSTEM

WORKING PAPER SERIES

NO 1608 / NOVEMBER 2013

OPTIMAL CONTROL WITH HETEROGENEOUS AGENTS IN CONTINUOUS TIME

Galo Nuño

In 2013 all ECB publications feature a motif taken from the €5 banknote.

Acknowledgements
The views expressed in this manuscript are those of the author and do not necessarily represent the views of the European Central Bank. The author is very grateful to Fernando Alvarez, Luca Dedola, Benjamin Moll, Alessio Moro, Giulio Nicoletti, Carlos Thomas and Oreste Tristani for helpful comments and suggestions. All remaining errors are mine.

Galo Nuño
European Central Bank; e-mail: galo.nuno@ecb.europa.eu

Address	Kaiserstrasse 29, 60311 Frankfurt am Main, Germany
Postal address	Postfach 16 03 19, 60066 Frankfurt am Main, Germany
Telephone	+49 69 1344 0
Internet	http://www.ecb.europa.eu
Fax	+49 69 1344 6000

ISSN	1725-2806 (online)
EU Catalogue No	QB-AR-13-105-EN-N (online)

This paper can be downloaded without charge from http://www.ecb.europa.eu or from the Social Science Research Network electronic library at http://ssrn.com/abstract_id=2347458.

Information on all of the papers published in the ECB Working Paper Series can be found on the ECB's website, http://www.ecb.europa.eu/pub/scientific/wps/date/html/index.en.html

Abstract

This paper introduces the problem of a planner who wants to control a population of heterogeneous agents subject to idiosyncratic shocks. The agents differ in their initial states and in the realization of the shocks. In continuous time, the distribution of states across agents is described by a Kolmogorov forward equation. The planner chooses the controls in order to maximize an optimality criterion subject to an "aggregate resource constraint". We demonstrate how the solution should satisfy a system of partial differential equations that includes a generalization of the Hamilton-Jacobi-Bellman equation and the Kolmogorov forward equation.

Keywords: Kolmogorov forward equation, calculus of variations, dynamic programming, heterogeneous agents.

JEL codes: C6, D3, D5, E2

Non-technical summary

Many problems of interest in economics and finance are composed of a very large number of agents. An economy, for example, is composed of millions of households and firms. A network may contain thousands of nodes. In these cases, assuming a distribution of state variables seems to be a reasonable approximation to the real problem under consideration.

The aim of this paper is to introduce optimal control problems in which there is an infinite number of heterogeneous agents. In optimal control, a planner deals with the problem of finding some control variables for a given system in order to achieve a specific goal. The state of the system is typically characterized by a number of state variables. In the case of a very large number of agents, assuming a distribution of state variables may seem a valid approximation to the real problem under consideration.

We assume that the state of each individual agent is characterized by a finite set of state variables. The evolution of the state variables of each agent is controllable, that is, there exist some controls that allow the planner to modify it. The evolution of the state variables is also subject to some random disturbances. In this problem, the aim of the planner is to maximize an optimality goal over the full distribution (across agents) of state variables.

This paper provides the mathematical foundations to solve optimal control problems with an infinite number of agents. In order to do so, it considers continuous time instead of the more standard assumption in macroeconomics of discrete time. The reason is that, in continuous time there is an equation (called Kolmogorov forward or Fokker-Planck equation) that describes the aggregate evolution of the distribution of state variables. This makes the problem more tractable and allows us to present the conditions to find a general solution to this kind of problems.

To illustrate the theory, we present a couple of examples. The first one is an optimal control version of the Aiyagari (1994) model. This is a model in which households own a non-negative amount of a single asset (capital) and provide a stochastic labor supply. A final good is produced by combining households' capital and labor. A social planner, following an utilitarian criterion, has to determine the income and consumption flows of households in order to maximize the average agents' utility, which depends on their consumption.

We introduce the problem and provide an analytical solution for the case of a stationary equilibrium, in which the capital distribution does not change with time. In the optimal allocation consumption is constant across households, independently of the household's capital holdings or of its labor supply. The income flow is just enough to cover for capital depreciation and consumption, so that there is no change in the capital of each individual household across time.

The second example is that of a monopolist in a model with heterogeneous reserves of an exhaustible resource. It extends previous work by Pindyck (1980) and Stiglitz (1976) to a heterogeneous-reserve setting. There is a distribution of deposits of an exhaustible commodity. The level of reserves in each deposit is known with certainty, but its instantaneous change is in part random due to an uncertainty component. The commodity can be extracted at a cost which depends on the extracted amount and on the level of reserves. We assume that all the deposits are owned by a risk-neutral producer. The producer observes the market demand for her commodity and has to decide the rate of extraction in each deposit given the reserve dynamics

In the solution to the problem, the monopolist applies a price markup equal to the inverse price demand elasticity. The individual production decision in each well is such that the price minus the markup equals the marginal extraction costs plus the scarcity rent, which differs in each deposit.

1 Introduction

Optimal control is an essential tool in economics and finance. In optimal control, a planner deals with the problem of finding a set control variables for a given system such that a certain optimality criterion is achieved. The state of the system is typically characterized by a finite number of state variables.[1] Some systems of interest are composed of a very large number of agents; an economy, for example, is composed of millions of households and firms and a network may contain thousands of nodes. In these cases, assuming a *distribution* of state variables seems to be a reasonable approximation to the real problem under consideration.

The aim of this paper is to introduce optimal control problems in which there is an infinite number of agents. The state of each of these agents is characterized by a finite set of state variables. The evolution of the state variables of each agent follows a controllable stochastic process, that is, there exists a vector of controls that allows the planner to modify the state of each agent. The evolution of the state variables is also subject to some random disturbances. In this problem, the aim of the planner is to maximize an optimality criterion over the full distribution (across agents) of state variables.

We focus on the continuous time version of the problem. In this case, each agent is characterized by a multidimensional *Itô process* with a drift coefficient that is a function of both its state and a *Markov control*. The control depends on time and on the current state of the agent. Randomness is introduced through an idiosyncratic Brownian motion. All the agents have the same drift and diffusion coefficients, but they differ in their initial states and in the realizations of the Brownian motion.

The key advantage of working in continuous time, compared to discrete time, is that the evolution of the state distribution across agents can be characterized by the *Kolmogorov forward* equation (also known as Fokker-Planck equation). This is a second-order partial differential equation (PDE) that describes the evolution of the distribution given its initial value and the control. Despite the random evolution of each individual agent, the dynamics of the distribution are deterministic. Thanks to this property of continuous time stochastic processes, the control of an infinite number of agents under idiosyncratic shocks can be expressed as the deterministic problem of controlling a distribution that evolves according to the Kolmogorov forward (KF) equation.

The optimality criterion is characterized by a *functional* of the state distribution and the control. In addition, the problem includes some subsidiary conditions, typically related with an *aggregate resource constraint.*

The main contribution of the paper is to present the necessary conditions for a solution to this problem. This can be done by following similar steps as in deterministic dynamic programming.

[1] See Bertsekas (2005, 2012) or Fleming and Soner (2006).

The difference is that, instead of working with functions using differential calculus, we should work with state distributions employing *calculus of variations*. The necessary conditions generalize the Hamilton-Jacobi-Bellman (HJB) equation in traditional dynamic programming. The role of the value function is played here by the *functional derivative* of the value functional with respect to the state distribution. If it exists and satisfies some regularity conditions, then it can be found, together with the optimal controls and the state distribution, as the solution of a coupled system of second-order PDEs. This system includes the generalization of the HJB equation, the KF equation, the aggregate resource constraint and the boundary conditions.

Lucas and Moll (2013) also analyze a problem of optimal control with heterogeneous agents. The main difference is that the present paper shows how to handle the resource constraints that are present in most economic problems, whereas the problem in Lucas and Moll is unconstrained. In our case, the presence of an aggregate constraint over the distribution of states and controls introduces an extra terms in the HJB equation weighted by a time-varying costate. The costate plays the same role as the Lagrange multipliers in standard constrained optimization, as it represents the marginal cost of violating the aggregate resource constraint.

To illustrate the theory, we present a couple of examples. The first one is a continuous time, optimal control version of the Aiyagari (1994) model. This is a stochastic growth model in which households own a non-negative amount of a single asset (capital) and provide a stochastic labor supply. A final good is produced by combining households' capital and labor. A social planner, following a *utilitarian* criterion, has to determine the income and consumption flows of households in order to maximize the average agents' utility, which depends on their consumption.

We introduce the problem and provide an analytical solution for the case of a stationary equilibrium, in which the capital distribution is time-invariant. In the optimal allocation consumption is constant across households, independently of the household's capital holdings or of its labor supply. The income flow is just enough to cover for capital depreciation and consumption, so that there is no change in the capital of each individual household across time. There are infinite distributions that satisfy the necessary conditions. The optimal stationary equilibrium is the same as the deterministic steady-state in the case of a representative household.

The second example is that of a monopolist in a model with heterogeneous reserves of an exhaustible resource. It extends previous work by Pindyck (1980) and Stiglitz (1976) to a heterogeneous-reserve setting. There is a distribution of deposits of an exhaustible commodity. The level of reserves in each deposit is known with certainty, but its instantaneous change is in part random due to an uncertainty component. The commodity can be extracted at a cost which depends on the extracted amount and on the level of reserves. We assume that all the deposits are owned by a risk-neutral producer. The producer observes the market demand for her commodity and has to decide the rate of extraction in each deposit given the reserve dynamics

In the solution to the problem, the monopolist applies a price markup equal to the inverse price demand elasticity. The individual production decision in each deposit is such that the price minus the markup equals the marginal extraction costs plus the scarcity rent, which differs in each deposit.

This paper is related to the large literature studying general equilibrium models with heterogeneous agents. Early contributions are Bewley (1986) and Aiyagari (1994). See Heathcote, Storesletten and Violante (2009) for a recent survey. Some existing papers analyze heterogeneous agent models in continuous time. Examples are Luttmer (2007) and Alvarez and Shimer (2011).[2] All these papers focus on the decentralized equilibrium. We contribute to this literature by developing a method to analyze the optimal allocation. The method is flexible enough to address issues such as different optimality critera or the optimal allocation in the transitional dynamics.

The structure of the paper is as follows. In section 2 we describe the problem. In section 3 we provide necessary conditions for a solution. In section 4 we present two examples. Finally, in section 5 we conclude.

2 Statement of the problem

Individual agents. We consider a continuous-time infinite-horizon economy. There is a continuum of unit mass of agents indexed by $i \in [0,1]$. Let X_t^i denote the state of an agent i at time $t \in \mathbb{R}_+$. The state evolves according to a multidimensional Itô process of the form

$$dX_t^i = b\left(X_t^i, u(t, X_t^i)\right) dt + \sigma\left(X_t^i\right) dB_t^i, \tag{1}$$

where $X_t^i \in Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $b : Q \times U \to Q$, $\sigma : Q \to Q$ and B_t^i is a n-dimensional Brownian motion.[3] Here $u : \mathbb{R}^+ \times Q \to U \subset \mathbb{R}^m$, $m \in \mathbb{N}$, is a *Markov control* whose value can be chosen in the given compact set U, and Q is a closed set. Assume that all elements outside of the diagonal in $\sigma\left(X_t^i\right)$ are zero. Notice that all the agents are identical in their drift and diffusion coefficients but potentially differ in their state and in the realization of the idiosyncratic Brownian motions.

[2] These papers are related to the emerging literature on *mean field games* (MFG) in mathematics. Introduced in Lasry and Lions (2007), MFG equilibria are a generalization of Nash equilibria in stochastic differential games as the number of players tend to infinite and agents only care about the distribution of other players' states. MFG are characterized by a HJB equation describing the value function of each player and a KF equation describing the evolution of the state distribution.See, for example, Guéant, Lasry and Lions (2011) or Carmona, Delarue and Lachapelle (2013)

[3] The set $\mathbb{R}^+$ includes all the non-negative real numbers, including zero. If zero is not included, we denote it as $\mathbb{R}^{++}$.

It is convenient to express $b(\cdot), u(\cdot)$ and $\sigma(\cdot)$ in matrix form

$$b\left(x, u(t,x)\right) = \begin{pmatrix} b_1\left(x_1,...,x_n; u_1(t,x_1,...,x_n),...,u_m(t,x_1,...,x_n),\right) \\ b_2\left(x_1,...,x_n; u_1(t,x_1,...,x_n),...,u_m(t,x_1,...,x_n),\right) \\ \vdots \\ b_n\left(x_1,...,x_n; u_1(t,x_1,...,x_n),...,u_m(t,x_1,...,x_n),\right) \end{pmatrix},$$

and

$$\sigma(x) = \begin{pmatrix} \sigma_{11}\left(x_1,...,x_n\right) & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & \sigma_{nn}\left(x_1,...,x_n\right) \end{pmatrix}.$$

To ensure the existence of a solution of the stochastic differential equation (1), assume that $b(x,u(t,x)) \in C^2(Q \times U)$ and $\sigma(x) \in C^2(Q)$ are measurable functions satisfying

$$|b(x,u)| + |\sigma(x)| \leq C_\gamma \left(1 + |x|\right);$$

$$|b(x,u) - b(y,u)| + |\sigma(x) - \sigma(y)| \leq D_\gamma \left(1 + |x-y|\right);$$

for all $x, y \in Q$, some constants C_γ and D_γ and $u \in U$ such that $|u| \leq \gamma$.[4] Also assume that $u(t,x)$ is adapted to the natural filtration of B_t^i.

A control law $u(t,x) \in U$ is *admissible* if for any initial point (t,x) such that $X_t^i = x$ the stochastic differential equation (1) has a unique solution. We denote A as the space of all bounded measurable controls contained in the set of all Markov controls $u(t,x) \in U$.

Aggregate distribution. Assume that the transition measure of X_t^i has a density $f(t,x;x_0^i)$: $\mathbb{R}^+ \times Q \to [0,1]$, ($f \in C^2(\mathbb{R}^+ \times Q)$), i.e., that

$$E\left[\varphi(X_t^i)|X_0^i = x_0\right] = \int_Q \varphi(x) f(t,x;x_0^i)dx; \qquad \forall \varphi \in C^2(Q),$$

the initial distribution of X_t^i at time $t=0$ is $\rho(x) \in C^2(Q)$.

In this case, the dynamics of the distribution of agents $f(t,x)$ are given by the KF equation[5]

$$\frac{\partial f(t,x)}{\partial t} = -\nabla_x \cdot \left[b\left(x, u(t,x)\right) f(t,x)\right] + \frac{1}{2}\Delta_x\left[\sigma^2(x) f(t,x)\right], \tag{2}$$

$$f(0,x) = \rho(x), \qquad \int_Q f(t,x)dx = 1, \tag{3}$$

[4] $|\sigma| = \sum_{i,j=1}^n \sigma_{ij}$. See Øksendal (2010) or Fleming and Soner (2006).

[5] Although the normalization of probability to one is not an initial condition, we always display it together with the initial condition for the sake of space.

where $\sigma^2 = \sigma\sigma'$. $\nabla_x \cdot$ is the *divergence operator* with respect to x, and Δ_x is the *Laplacian operator* with respect to x.

Performance functional. Let $G[f,u]$ be a differentiable functional and $r > 0$ is a constant. We define the *performance functional* $J[f(t,\cdot),u]$ as

$$J[f(t,\cdot),u] \equiv \int_t^\infty e^{-r(s-t)} G[f(s,\cdot),u]\,ds.$$

We also define set of constraints that should be satisfied

$$H[f(t,\cdot),u] = 0, \tag{4}$$

where H is the *resource constraint functional.* The reason of this name will be clear in the examples below. This is a family of constraints indexed by time.

Definition 1 (optimal control with heterogeneous agents) *The optimal control problem with heterogeneous agents is to find, for each $f \in C^2(Q)$ that satisfies the Kolmogorov forward equation (2, 3) and $t \in \mathbb{R}^+$, a number $V[f(t,\cdot)]$ and a control $u^* = u^*(t,x) \in A$ such that*

$$V[f(t,\cdot)] \equiv \sup_{u(\cdot)\in A} J[f(t,\cdot),u] = J[f(t,\cdot),u^*] \tag{5}$$

subject to the resource constraint functional (4).

The supremum is taken over a family A of admissible controls. Such control is called an *optimal control* and $V[f(t,\cdot)]$ is the *value functional.* The optimal control problem with heterogeneous agents is an extension of the classical deterministic optimal control problem to an infinite dimensional setting, in which the state is the whole distribution of individual states $f(t,x)$. The key point is that the law of motion of this distribution is given by the KF equation.

Finally, we assume that

$$\lim_{t\uparrow\infty} e^{-rt} V[f(t,\cdot)] = 0. \tag{6}$$

3 Solution to the problem

In this section we characterize the solution to problem (5). First, we introduce the dynamic programming principle for this problem.

Lemma 2 (Dynamic programming) *For any initial condition $f(t_0,x)$ such that $t_0 \in \mathbb{R}^+$, $f(t_0,x) \in C^2(Q)$, suppose that the admissible control $u^* \in A$ is a solution to the problem (5)* *for $t_0 \leq t < \infty$, then*

$$V\left[f(t_0,\cdot)\right] = \int_{t_0}^{t} e^{-r(s-t_0)} G\left[f(s,),u^*\right] ds + e^{-r(t-t_0)} V\left[f(t,\cdot)\right], \tag{7}$$

Proof. This lemma is a particular case of lemma 7.1 in Flemming and Soner (2006) replacing the value function by $V\left[f(t_0,\cdot)\right]$. ■

We can define

$$v(t,x) \equiv \frac{\delta V\left[f\right]}{\delta f(t,x)}, \tag{8}$$

as the functional derivatives of the value functional $V\left[f\right]$ with respect to f at point (t,x), $v : \mathbb{R}^+ \times Q \to \mathbb{R}$.[6] Lucas and Moll (2013) provide an economic interpretation of $v(t,x)$ as the *marginal social value* at time t of an agent in state x.

Given (8), we provide necessary conditions to the problem.

Theorem 3 (Necessary conditions) *Assume that $v \in C^2([0,\infty)\times Q)$ and that an optimal Markov control u^* exists. Then, it satisfies*

$$\frac{\delta G\left[f,u^*\right]}{\delta f(t,x)} + \lambda(t)\frac{\delta H\left[f,u^*\right]}{\delta f(t,x)} + \frac{\partial v(t,x)}{\partial t} + \sum_{i=1}^{n} b_i\left(x,u^*\right)\frac{\partial v(t,x)}{\partial x_i} + \sum_{i=1}^{n}\frac{\sigma_{ii}^2(x)}{2}\frac{\partial^2 v(t,x)}{\partial x_i^2} = rv(t,x) \tag{9}$$

$$\frac{\delta G\left[f,u^*\right]}{\delta u_j(t,x)} + \lambda(t)\frac{\delta H\left[f,u^*\right]}{\delta u_j(t,x)} + \sum_{i=1}^{n}\frac{\partial b_i\left(x,u^*\right)}{\partial u_j}\frac{\partial v(t,x)}{\partial x_i} f(t,x) = 0, \tag{10}$$

$j = 1,...,m$, with $\lambda(t) : \mathbb{R}^+ \to \mathbb{R}$, together with the KF equations (2, 3), the resource constraint (4) and the transversality condition

$$\lim_{t\uparrow\infty} e^{-rt} v(t,x) = 0. \tag{11}$$

Proof. Suppose that $0 < t < \infty$. If an optimal Markov control $u^* = [u_1^*,...,u_m^*]$ exists and satisfies the constraint (4), $H[f(t,\cdot),u^*] = 0$, it should be an extremal of

$$\int_{t}^{\infty} e^{-r(s-t)}\left\{G\left[f(s,\cdot),u\right] + \lambda(s)H[f(s,\cdot),u]\right\} ds,$$

[6] For an introduction to the Calculus of Variations, see Gelfand and Fomin (1991).

where $\lambda(t): \mathbb{R}^{+} \rightarrow \mathbb{R}$.[7] Taking derivatives with respect to time in equation (7):

$$\begin{aligned} rV\left[f(t,\cdot)\right] &= G\left[f(t,\cdot),u^{*}\right]+\frac{\partial}{\partial t}V\left[f(t,\cdot)\right]=G\left[f(t,\cdot),u^{*}\right]+\int_{Q}\frac{\delta V\left[f\right]}{\delta f(t,y)}\frac{\partial f(t,y)}{\partial t}dy \\ &= G\left[f(t,\cdot),u^{*}\right]+\int_{Q}\frac{\delta V\left[f\right]}{\delta f(t,y)}\left(-\nabla_{y}\cdot\left[b\left(y,u^{*}(t,y)\right)f(t,y)\right]+\frac{1}{2}\Delta_{y}\left[\sigma^{2}(y)f(t,y)\right]\right)dy. \end{aligned}$$

The optimal Markov control $u^{*}(t,\cdot)$ that maximizes the performance functional $J\left[f(t,\cdot),u\right]$ (5) subject to the constraints (4), $H[f(t,\cdot),u]=0$, should be an extremal of

$$G\left[f(t,\cdot),u\right]+\lambda(t)H[f(t,\cdot),u]+\int_{Q}v(t,y)\left(-\nabla_{y}\cdot\left[b\left(y,u(t,y)\right)f(t,y)\right]+\frac{1}{2}\Delta_{y}\left[\sigma^{2}(y)f(t,y)\right]\right)dy, \tag{12}$$

where $v(t,y)=\frac{\delta V[f]}{\delta f(t,y)}$ and $\lambda(t): \mathbb{R}^{+} \rightarrow \mathbb{R}$. Thus, the functional derivative of (12) with respect to u_j, $j=1,...,m$, should be zero:

$$\frac{\delta G\left[f,u^{*}\right]}{\delta u_{j}(t,x)}+\lambda(t)\frac{\delta H\left[f,u^{*}\right]}{\delta u_{j}(t,x)}+\frac{\delta}{\delta u_{j}(t,x)}\int_{Q}v(t,y)\left(-\nabla_{y}\cdot\left[b\left(y,u^{*}(t,y)\right)f(t,y)\right]+\frac{1}{2}\Delta_{y}\left[\sigma^{2}(y)f(t,y)\right]\right)dy=0.$$

Notice that

$$\nabla_{x}\cdot\left[b\left(x,u(t,x)\right)f(t,x)\right]=\sum_{i=1}^{n}\left[\frac{\partial b_{i}}{\partial x_{i}}f+\sum_{j=1}^{m}\frac{\partial b_{i}}{\partial u_{j}}\frac{\partial u_{j}}{\partial x_{i}}f+b_{i}\frac{\partial f}{\partial x_{i}}\right].$$

Also notice that, in calculus of variations, if

$$\Phi[u_{1},...,u_{m}]=\int_{Q}\varphi(x,u_{1},...,u_{m},\sum_{i=1}^{n}\frac{\partial u_{1}}{\partial x_{i}},...,\sum_{i=1}^{n}\frac{\partial u_{m}}{\partial x_{i}})dx,$$

with $u_j: Q \rightarrow U_j \subset \mathbb{R}$, then

$$\frac{\delta\Phi}{\delta u_{j}}=\frac{\partial\varphi}{\partial u_{j}}-\sum_{i=1}^{n}\frac{\partial}{\partial x_{i}}\frac{\partial\varphi}{\partial\left(\frac{\partial u_{j}}{\partial x_{i}}\right)}.$$

Therefore,

$$\begin{aligned} &\frac{\delta}{\delta u_{j}(t,x)}\int_{Q}v(t,y)\left(-\nabla_{y}\cdot\left[b\left(y,u(t,y)\right)f(t,y)\right]+\frac{1}{2}\Delta_{y}\left[\sigma^{2}(y)f(t,y)\right]\right)dy \\ &= -\sum_{i=1}^{n}\left[\frac{\partial^{2}b_{i}}{\partial u_{j}\partial x_{i}}vf+\sum_{k=1}^{m}\frac{\partial^{2}b_{i}}{\partial u_{j}\partial u_{k}}\frac{\partial u_{k}}{\partial x_{i}}vf+\frac{\partial b_{i}}{\partial u_{j}}\frac{\partial f}{\partial x_{i}}v\right]+\sum_{i=1}^{n}\frac{\partial}{\partial x_{i}}\left(vf\frac{\partial b_{i}}{\partial u_{j}}\right) \\ &= \sum_{i=1}^{n}\frac{\partial b_{i}}{\partial u_{j}}\frac{\partial v}{\partial x_{i}}f, \end{aligned}$$

[7] See Gelfand and Fomin (1993, pp. 42-46).

so we obtain equation (10). We can also take the functional derivative with respect to f in equation (12) noticing that $u^* = u[f]$, that is, the optimal control depends on the state distribution:

$$\begin{aligned}
&\frac{\delta G\left[f,u^*\right]}{\delta f(t,x)} + \lambda(t)\frac{\delta H\left[f,u^*\right]}{\delta f(t,x)} \\
&+\frac{\delta}{\delta f(t,x)}\int_Q \frac{\delta V\left[f\right]}{\delta f(t,y)}\left(-\nabla_y\cdot\left[b\left(y,u(t,y)\right)f(t,y)\right] + \frac{1}{2}\Delta_y\left[\sigma^2(y)f(t,y)\right]\right)dy \\
&+\sum_{j=1}^m \overbrace{\frac{\delta G\left[f,u^*\right]}{\delta u_j(t,x)} + \lambda(t)\frac{\delta H\left[f,u^*\right]}{\delta u_j(t,x)} + \frac{\delta}{\delta u_j(t,x)}\int_Q w\left(-\nabla_y\cdot\left[bf\right] + \frac{1}{2}\Delta_y\left[\sigma^2 f\right]\right)dy}^{0} = 0,
\end{aligned}$$

where the third line is zero due to the optimality of the controls. Notice that

$$\begin{aligned}
\frac{1}{2}\Delta_x\left[\sigma^2(x)f(t,x)\right] &= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left[\sigma_{ii}\frac{\partial\sigma_{ii}}{\partial x_i}f + \frac{\sigma_{ii}^2}{2}\frac{\partial f}{\partial x_i}\right] \\
&= \sum_{i=1}^n\left[\left(\frac{\partial\sigma_{ii}}{\partial x_i}\right)^2 f + \sigma_{ii}\frac{\partial^2\sigma_{ii}}{\partial x_i^2}f + 2\sigma_{ii}(x)\frac{\partial\sigma_{ii}}{\partial x_i}\frac{\partial f}{\partial x_i} + \frac{\sigma_{ii}^2}{2}\frac{\partial^2 f}{\partial x_i^2}\right],
\end{aligned}$$

then,

$$\begin{aligned}
&\frac{\delta}{\delta f(t,x)}\int_Q \frac{\delta V\left[f\right]}{\delta f(t,y)}\left(-\nabla_y\cdot\left[b\left(y,u(t,y)\right)f(t,y)\right] + \frac{1}{2}\Delta_y\left[\sigma^2(y)f(t,y)\right]\right)dy \\
=& \int_Q \frac{\delta^2 V\left[f\right]}{\delta f(t,x)\delta f(t,y)}\frac{\partial f}{\partial t}dy + \sum_{i=1}^n\left[-\frac{\partial b_i}{\partial x_i} - \sum_{j=1}^m\frac{\partial b_i}{\partial u_j}\frac{\partial u_j}{\partial x_i} + \left(\frac{\partial\sigma_{ii}}{\partial x_i}\right)^2 + \sigma_{ii}(x)\frac{\partial^2\sigma_{ii}}{\partial x_i^2}\right]v \\
&-\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(-b_i v + 2\sigma_{ii}\frac{\partial\sigma_{ii}}{\partial x_i}v\right) + \sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}\left(2\sigma_{ii}\frac{\partial\sigma_{ii}}{\partial x_i}\frac{\sigma_{ii}^2}{2}v\right) \\
=& \int_Q \frac{\delta^2 V\left[f\right]}{\delta f(t,x)\delta f(t,y)}\frac{\partial f}{\partial t}dy + \sum_{i=1}^n b_i\frac{\partial v}{\partial x_i} + \sum_{i=1}^n\frac{\sigma_{ii}^2}{2}\frac{\partial^2 v}{\partial x_i^2},
\end{aligned}$$

and taking into account that

$$\begin{aligned}
\frac{\partial v(t,x)}{\partial t} &= \frac{\partial}{\partial t}\frac{\delta V\left[f\right]}{\delta f(t,x)} = \int_Q \frac{\delta^2 V\left[f\right]}{\delta f(t,y)\delta f(t,x)}\frac{\partial f(t,y)}{\partial t}dy \\
&= \int_Q \frac{\delta^2 V\left[f\right]}{\delta f(t,x)\delta f(t,y)}\frac{\partial f(t,y)}{\partial t}dy,
\end{aligned}$$

we obtain equation (9). Finally, suppose that $t\uparrow\infty$. In this case, using (6) and taking functional derivatives with respect to f we obtain (11). ■

Theorem 4 provides necessary conditions for the existence of an optimal control. It provides a system of $m+2$ PDEs that link the dynamics of $v(t,x)$, $u^*(t,x)$ and $f(t,x)$. Equations (9) and (10) are a generalization of the HJB equation to our particular infinite-dimensional problem. The system is forward-backward and coupled in the sense that the HJB is a function of the distribution f and has boundary conditions at time $t \uparrow \infty$, whereas the KF is a function of the optimal controls u^* and it has boundary conditions at time $t=0$.

4 Examples

4.1 Optimal control in an Aiyagari-Bewley economy

We introduce as an example an optimal control problem in a stochastic growth model with idiosyncratic shocks *à la* Aiyagari (1994) or Bewley (1986).

Statement of the problem. Assume a population of households of total mass 1. Each household i receives an income $y_t^i \in \mathbb{R}$ and consumes an amount $c_t^i > 0$. Notice that the income may be negative, reflecting the possibility that the household provides resources to other households. The household may own a single asset, capital, in a *non-negative amount* $k_t^i > 0$,[8] according to

$$dk_t^i = \left(y_t^i - \delta k_t^i - c_t^i\right) dt,$$

where $\delta \in [0,1]$ is the rate of depreciation. Each household may contribute to the aggregate production of goods with its capital stock and its labor supply. The labor supply is assumed to follow an Itô process

$$dl_t^i = \mu_L(l_t^i)dt + \sigma_L(l_t^i)dB_t^i,$$

where $l_t^i \in [0,1]$, B_t^i is a Brownian motion independent across agents and $\mu_L, \sigma_L : [0,1] \to \mathbb{R}$ are $C^2([0,1])$. Total labor is constant $L_t = 1/2$.

Assume that there is a single good Y_t that can be used for consumption and investment. It is produced using a Cobb-Douglas function of aggregate capital input, K_t, and aggregate labor input L_t:

$$Y_t = zK_t^{\alpha} L_t^{1-\alpha}, \tag{13}$$

with $\alpha \in [0,1]$ and $z > 0$.

Let $f(t,k,l) \in C^2(\mathbb{R}^+ \times \mathbb{R}^+ \times [0,1])$ be the joint distribution of capital and labor supply. It

[8] In a competitive equilibrium, the non-negativity constraint in capital holdings is typically known as a *borrowing constraint*.

evolves according to the KF equation

$$\frac{\partial f(t,k,l)}{\partial t} = -\frac{\partial}{\partial k}\left\{\left[y^*(t,k,l) - \delta k - c^*(t,k,l)\right] f(t,k,l)\right\} \tag{14}$$

$$-\frac{\partial}{\partial l}\left[\mu_L(l) f(t,k,l)\right] + \frac{1}{2}\frac{\partial^2}{\partial l^2}\left[\sigma_L^2(l) f(t,k,l)\right], \tag{15}$$

where $y^*(t,k,l)$ and $c^*(t,k,l)$ are the optimal income and consumption. The joint initial distribution is

$$f(0,k,l) = \rho(k,l) \in C^2(\mathbb{R}^+ \times [0,1]), \qquad \int_0^\infty \int_0^1 f(t,k,l)dldk = 1. \tag{16}$$

The aggregate capital is defined as

$$K(t) \equiv \int_0^\infty \int_0^1 kf(t,k,l)dldk. \tag{17}$$

Assume that a social planner maximizes the *average* utility across the population, that is,

$$G[f,y,c] = \int_0^\infty \int_0^1 U(c(t,k,l))f(t,k,l)dldk, \tag{18}$$

where $U(c) \in C^2(\mathbb{R}^+)$ is the instantaneous individual utility of an individual that consumes a flow of c units. U is assumed to be a strictly increasing and strictly concave function. The social planner discounts the future exponentially with discount rate $r > 0$. The performance functional $J\left[f(t,\cdot),y,c\right]$ is

$$J\left[f(t,\cdot),y,c\right] \equiv \int_t^\infty e^{-r(s-t)}\left[\int_0^\infty \int_0^1 U(c(t,k,l))f(t,k,l)dldk\right] ds$$

and the resource constraint condition

$$H[f(t,\cdot),y,c] \equiv zK_t^\alpha L_t^{1-\alpha} - \int_0^\infty \int_0^1 y(t,k,l)f(t,k,l)dldk = 0, \tag{19}$$

that is, aggregate production is redistributed across agents through the income flows.

We define the optimal control problem:

Definition 4 (optimal control in the Aiyagari model) *The optimal control problem is to find, for each $f \in C^2(\mathbb{R}^+ \times \mathbb{R}^+ \times [0,1])$ that satisfies the Kolmogorov forward equation (14, 16) and $t \in \mathbb{R}^+$, a number $V\left[f(t,\cdot)\right]$ and a set of controls $\{y^*(t,k,l), c^*(t,k,l)\} \in A$ such that*

$$V\left[f(t,\cdot)\right] \equiv \sup_{\{y^*,c^*\}\in A} J\left[f,y,c\right] \tag{20}$$

subject to the resource constraint functional (19).

The supremum is taken again over a family A of admissible controls.

Necessary conditions. Applying the results of Theorem 3, in order to compute the necessary conditions of the problem we need the functional derivatives with respect to the distribution and the controls

$$\begin{array}{ll} \frac{\delta G}{\delta f} = U(c(t,k,l)), & \frac{\delta H}{\delta f} = \alpha \frac{Y_t}{K_t} k + (1-\alpha)\frac{Y_t}{L_t} l - y, \\ \frac{\delta G}{\delta y} = 0, & \frac{\delta H}{\delta y} = -f(t,k,l), \\ \frac{\delta G}{\delta c} = \frac{\partial U(c(t,k,l))}{\partial c} f(t,k,l), & \frac{\delta H}{\delta c} = 0, \end{array}$$

and thus equations (9) and (10) can be expressed together as a HJB equation

$$\begin{aligned} rv(t,k,l) &= \sup_{\{y^*,c^*\}\in A} U(c(t,k,l)) + \lambda(t)\left[\alpha z K^{\alpha-1}(t) L^{1-\alpha}(t) k + (1-\alpha) z K^{\alpha}(t) L^{-\alpha}(t) l - y(t,k,l)\right] \\ &+ \frac{\partial v(t,k,l)}{\partial t} + (y(t,k,l) - \delta k - c(t,k,l))\frac{\partial v(t,k,l)}{\partial k} \\ &+ \mu_L(l)\frac{\partial v(t,k,l)}{\partial l} + \frac{\sigma_L^2(l)}{2}\frac{\partial^2 v(t,k,l)}{\partial l^2}, \end{aligned}$$

plus the KF equation (14), the initial distribution (16), the transversality condition $\lim_{t\uparrow\infty} e^{-rt} v(t,k,l) = 0$, and the definition of the resource constraint (19).

A particular case is that of a *stationary equilibrium,* in which the distribution $f(t,k,l)$ is stationary ($\frac{\partial f(t,k,l)}{\partial t} = 0$) and there is no dependence with respect to time. We analyze the stationary equilibrium for the case of $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma \in (0,1)$, $\mu_L(l) = 0$ and $\sigma_L^2(l) = \sigma^2$.

Proposition 5 *Assume a distribution $\rho(k)$ that satisfies*

$$\int_0^{\infty} k\rho(k)dk = \frac{1}{2}\left(\frac{\alpha z}{r+\delta}\right)^{\frac{1}{1-\alpha}}, \tag{21}$$

then (a) the distribution $f(t,k,l) = \rho(k)$ is a stationary equilibrium of the problem; (b) the aggregate capital is $K = \frac{1}{2}\left(\frac{\alpha z}{r+\delta}\right)^{\frac{1}{1-\alpha}}$; (c) the Lagrange multiplier $\lambda = \left[\frac{(r+(1-\alpha)\delta}{\alpha} K\right]^{-\gamma}$; and (d) the optimal controls are $c^* = \frac{[r+(1-\alpha)\delta]}{\alpha} K$ and $y^*(k) = \delta k + c^*$.

Proof. See the Appendix. ■

Proposition 5 shows the optimal allocation in the case of a stationary equilibrium. There are infinite distributions that satisfy (21). In the optimal allocation consumption is constant across households, independently of the household's capital holdings or of its labor supply. The income flow is just enough to cover for capital depreciation and consumption, so that there is no change in the capital of each individual household across time ($dk_t^i = 0$). Notice that both consumption and income depend on $\{\alpha, z, r, \delta\}$, but not on the households' risk aversion γ or the volatility of the labor supply σ. The stationary allocation is the same as the deterministic steady-state in the case of a representative household.

4.2 A monopoly with heterogeneous exhaustible resources

The next example is the case of a monopolist in a model with heterogeneous reserves of an exhaustible resource. It extends previous work by Pindyck (1980) and Stiglitz (1976) to a heterogeneous setting.

Statement of the problem. There is a continuum of unit mass of "wells" or "mines" of a commodity indexed by $i \in [0, 1]$. Let $x_t^i \geq 0$ denote the commodity reserves at time t in a well i. Reserves evolve according to the stochastic process

$$dx_t^i = -q_t^i dt + \sigma x_t^i B_t^i, \tag{22}$$

where q_t^i is the rate of extraction at well i and B_t^i is a Brownian motion, $\sigma > 0$. Equation (22) implies that the current stock is known with certainty, but the instantaneous change in the stock is in part random due to the uncertainty component $\sigma x_t^i B_t^i$. The extraction rate $q_t^i = q(t, x_t^i) \geq 0$ is is an admissible Markov control.

The commodity can be extracted at a cost $C(q, x)$ which depends on the extracted amount and on the level of stocks. We assume that $\frac{\partial C(q,x)}{\partial q} > 0$ and $\frac{\partial C(q,x)}{\partial x} < 0$.

The distribution of reserves is given by the KF equation. Let $f(t, x)$ be the fraction of wells with reserves x at time t. The distribution dynamics are

$$\frac{\partial f(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left[-q(t,x)f(t,x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[(\sigma x)^2 f(t,x)\right],$$

reflecting how the evolution of the distribution depends on the extraction policy function $q(x, t)$. The initial distribution of reserves is $f(0, x) = \rho(x)$.

We assume that all the wells are owned by a risk-neutral producer. The producer observes the market demand for her commodity, $D(p)$, with $D'(p) < 0$,and has to decide the rate of extraction $q(t, x)$ *in each well* given the reserve dynamics (22). The monopolist also chooses the price $p(t, x)$, given the aggregate demand

$$p(t,x) = \bar{p}(t) = D^{-1}\left[\int_0^{\infty} q(t,x)f(t,x)dx\right],$$

where $D^{-1}(\cdot)$ is the inverse of D. Even if the monopolist might, in principle, choose different prices in each well, the centralised market structure forces her to set them all equal.

The producer maximizes total profits

$$G\left[f,q,p\right]=\int_0^\infty \left[p(t,x)q(t,x)-C(q(t,x),x)\right]f(t,x)dx.$$

These profits are discounted exponentially using the discount rate $r>0$. The performance functional is

$$J\left[f(t,\cdot),q,p\right]=\int_t^\infty e^{-r(s-t)}\left\{\int_0^\infty \left[p(t,x)q(t,x)-C(q(t,x),x)\right]f(t,x)dx\right\}ds,$$

and the resource constraint

$$H\left[f(t,\cdot),q,p\right]=\bar{p}(t)-D^{-1}\left[\int_0^\infty q(t,x)f(t,x)dx\right].$$

Necessary conditions. Applying Theorem 3 and introducing the costate $\lambda(t)$, the necessary conditions in this case are

$$\begin{aligned} rV(t,x) &= p^*(t,x)q^*(t,x)-C(q^*(t,x),x)-\lambda(t)\frac{q^*(t,x)}{D'(p)}-q^*(x,t)\frac{\partial V(t,x)}{\partial x} \\ & \frac{(\sigma x)^2}{2}\frac{\partial^2 V(t,x)}{\partial x^2}+\frac{\partial V(t,x)}{\partial t}, \end{aligned}$$

where $p^*(t,x)$ and $q^*(t,x)$ are the optimal controls, and

$$p^*(t,x)-\frac{\partial C(q^*,x)}{\partial q}-\lambda(t)\frac{1}{D'(p)} = \frac{\partial V(t,x)}{\partial x}, \tag{23}$$

$$q^*(t,x)f(t,x) = -\lambda(t)\bar{p}(t)\delta\left[x-x^{-1}(\bar{p}(t))\right], \tag{24}$$

where $\delta(x)$ is Dirac delta. To obtain (24) we have use the fact that $\bar{p}(t)=\int_0^\infty p(t,x)\delta\left[x-x^{-1}(\bar{p}(t))\right]dx$.

In order to obtain the value of the costate we integrate (24)

$$\int_0^\infty q^*(t,x)f(t,x)dx=D(p)=\int_0^\infty -\lambda(t)\bar{p}(t)\delta\left[x-x^{-1}(\bar{p}(t))\right]dx=-\lambda(t)\bar{p}(t),$$

so that $\lambda(t)=-\frac{D(\bar{p}(t))}{\bar{p}(t)}$. Therefore, as in the previous example, the problem can be expressed as a HJB equation

$$\begin{aligned} rV(t,x) &= \max_q \bar{p}(t)\left[1-\varepsilon(\bar{p}(t))\right]q(t,x)-C(q(t,x),x)-q(x,t)\frac{\partial V(t,x)}{\partial x} \\ & \frac{(\sigma x)^2}{2}\frac{\partial^2 V(t,x)}{\partial x^2}+\frac{\partial V(t,x)}{\partial t}, \end{aligned}$$

in which the monopolist is applying a price markup equal to the inverse price demand elasticity $\varepsilon(\bar{p}(t))=-\frac{D(\bar{p}(t))}{\bar{p}(t)D'(\bar{p}(t))}$.

The individual production decision in each well (23) is such that the price minus the markup ($\bar{p}(1-\varepsilon)$) equals the marginal extraction costs ($\frac{\partial C}{\partial q}$) plus the scarcity rent ($\frac{\partial V(t,x)}{\partial x}$).

5 Conclusion

This paper introduces the problem of a planner who tries to control a population of heterogeneous agents subject to idiosyncratic shocks in order to maximize an optimality criterion related to the distribution of states across agents. If the problem is analyzed in continuous time, the KF equation provides a deterministic law of motion of the entire distribution of state variables across agents. The problem can thus be analyzed as a deterministic optimal control in which both the control and the state are distributions. We provide necessary conditions by combining dynamic programming with calculus of variations. If a solution to the problem exists and satisfies some differentiability conditions, we show how it should satisfy a system of PDEs including a generalization of the HJB equation and a KF equation.

In this paper we provide an analytical solution for the example of social planning in a heterogeneous-agent economy. Unfortunately, the solution of most problems of interest will require the use of numerical methods. In this respect, we expect that the application of new numerical techniques such as the one introduced in Achdou et al. (2013) will allow solving more interesting and complex problems.

References

[1] Achdou, Y., J.-M. Lasry, P.-L. Lions and B. Moll (2013), "Heterogeneous Agent Models in Continuous Time," mimeo.

[2] Alvarez, F., and R. Shimer (2011), “Search and Rest Unemployment.” *Econometrica*, 79 (1), pp. 75–122.

[3] Aiyagari, R., (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109 (3), pp. 659-84.

[4] Bertsekas, D. P. (2005), *Dynamic Programming and Optimal Control*, Vol. I Athena Scientific, Belmont, MA.

[5] Bertsekas, D. P. (2012), *Dynamic Programming and Optimal Control*, Vol. II Athena Scientific, Belmont, MA.

[6] Bewley,T. (1986) "Stationary Monetary Equilibrium With a Continuum of Independently Fluctuating Consumers," in *Contributions to Mathematical Economics in Honor of Gérard Debreu*, ed. By W. Hildenbrand, and A. Mas-Colell. North Holland, Amsterdam.

[7] Carmona, R., F. Delarue and A. Lachapelle (2013), "Control of McKean–Vlasov Dynamics versus Mean Field Games," *Mathematics and Financial Economics*, 7 (2), pp. 131-166.

[8] Fleming, W. H. and H. M. Soner (2006), *Controlled Markov Processes and Viscosity Solutions*, Springer, Berlín.

[9] Gelfand, I. M. and S. V. Fomin (1991), *Calculus of Variations*, Dover Publications, Mineola, NY.

[10] Guéant, O., J.-M. Lasry and P.-L. Lions (2011), "Mean Field Games and Applications," *Paris-Princeton Lectures on Mathematical Finance*, 2011.

[11] Heathcote, J., K. Storesletten, and G. L. Violante (2009), "Quantitative Macroeconomics with Heterogeneous Households," *Annual Review of Economics*,1(1), pp. 319–354.

[12] Lasry, J.-M. and P.-L. Lions (2007), "Mean Field Games," *Japanese Journal of Mathematics*, 2 (1), pp. 229-260.

[13] Lucas, R. and B. Moll (2013), "Knowledge Growth and the Allocation of Time," *Journal of Political Economy*, forthcoming.

[14] Luttmer, E. G. J. (2007), "Selection, Growth, and the Size Distribution of Firms," *The Quarterly Journal of Economics*, 122 (3), pp. 1103–1144.

[15] Øksendal, B. (2010), *Stochastic Differential Equations: An Introduction with Applications*, Springer, Berlín.

[16] Pindyck, R. S. (1980), "Uncertainty and Exhaustible Resource Markets," *Journal of Political Economy*, 88 (6), pp. 1203-25.

[17] Stiglitz, J. E. (1976), "Monopoly and the Rate of Extraction of Exhaustible Resources," *American Economic Review*, 66 (4), pp. 655-61.

Appendix: Proof of proposition 5

Proof. We propose as an *ansatz* to problem (20) the optimal controls $c^*(k,l) = c^* = \frac{[r+(1-\alpha)\delta]}{\alpha}K$ and $y^*(k,l) = \delta k + c^*$ and the stationary distribution $f(t,k,l) = \rho(k)$, $\forall t \in \mathbb{R}^+$. The KF equation in that case is verified:

$$\frac{\partial \rho(k)}{\partial t} + \frac{\partial}{\partial k}\left\{\left[y^*(t,k,l) - \delta k - c^*(t,k,l)\right]\rho(k)\right\} - \frac{\sigma^2}{2}\frac{\partial^2 \rho(k)}{\partial l^2} = 0.$$

The aggregate capital is

$$K = \int_0^\infty k\rho(k)dk.$$

The first-order conditions are such that

$$\lambda = \frac{\partial v(k,l)}{\partial k},$$

so $v(k,l) = \lambda k + \phi(l)$, with $\phi \in C^2([0,1])$, and

$$\lambda = \frac{\partial U(c(k,l))}{\partial k} = (c^*)^{-\gamma}.$$

The HJB equation results in

$$r\left[\lambda k + \phi(l)\right] = U(c^*) + \lambda\left[\alpha z K^{\alpha-1}L^{1-\alpha}k + (1-\alpha)zK^{\alpha}L^{-\alpha}l - \delta k - c^*\right] + \frac{\sigma^2}{2}\frac{\partial^2 \phi(l)}{\partial l^2},$$

or equivalently,

$$\begin{aligned} r\lambda k &= \left[\alpha z K^{\alpha-1}L^{1-\alpha} - \delta\right]\lambda k, \\ 0 &= \frac{\sigma^2}{2}\frac{\partial^2 \phi(l)}{\partial l^2} - r\phi(l) + \lambda\left[(1-\alpha)zK^{\alpha}L^{-\alpha}l\right] + \left[U(c^*) - \lambda c^*\right]. \end{aligned}$$

The first expression can be rearranged as $K = \frac{1}{2}\left(\frac{\alpha z}{r+\delta}\right)^{\frac{1}{1-\alpha}}$. The second expression is a second-order ordinary differential equation to find $\phi(l)$. Finally, the market clearing condition is

$$\begin{aligned} \int_0^\infty \left(\delta k + c^*\right)dk &= zK^{\alpha}L^{1-\alpha}, \\ \lambda^{-1/\gamma} &= zK^{\alpha}L^{1-\alpha} - \delta K, \\ \lambda &= \left[\frac{(r+(1-\alpha)\delta)}{\alpha}K\right]^{-\gamma}. \end{aligned}$$

Therefore, the ansatz satisfies all the necessary conditions. ■